

Applications of Quotient Posets

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Abstract

In this paper we consider the characteristic polynomial of not necessarily ranked posets. We do so by allowing the rank to be an arbitrary function from the poset to the nonnegative integers. We will prove two results showing that the characteristic polynomial of a poset has nonnegative integral roots. Our factorization theorems will then be used to show that any interval of the Tamari lattice has a characteristic polynomial which factors in this way. Blass and Sagan's result about LL lattices will also be shown to be a consequence of our factorization theorems. Finally we will use quotient posets to give unified proofs of some classic Möbius function results.

1 Introduction

All the posets we will consider here will be finite and contain a minimum element which will be denoted by $\hat{0}$. Our focus will be on the (one-variable) Möbius function and its generating function, the characteristic polynomial. In particular, we will give some new theorems about when the characteristic polynomial of a poset has nonnegative integer roots. Additionally, we will introduce a new method for proving some of the classic results about the Möbius function. We begin with a review of the Möbius function and the characteristic polynomial.

Here we will use \mathbb{Z} to denote the set of integers and \mathbb{N} to be the set of nonnegative integers. Given a poset P , the Möbius function $\mu : P \rightarrow \mathbb{Z}$ is defined as the unique function on P such that

$$\sum_{y \leq x} \mu(y) = \delta_{\hat{0}, x}$$

where $\delta_{\hat{0}, x}$ is the Kronecker delta function.

We say a poset, P , is *ranked* if, for each $x \in P$, every saturated $\hat{0}$ - x chain has the same length. If a poset is ranked, the *rank function* $\rho : P \rightarrow \mathbb{N}$ is given by setting $\rho(x)$ to be the length of a $\hat{0}$ - x chain.

In the standard definition of the characteristic polynomial, one must have a ranked poset. However, to enlarge the set of posets we can consider, we will instead replace the rank function with any map $\rho : P \rightarrow \mathbb{N}$. While ρ is arbitrary here, certain conditions may be imposed on the function by the hypotheses of the various theorems considered later.

Given a $\rho : P \rightarrow \mathbb{N}$ we will define the *rank* of the poset as

$$\rho(P) = \max_{x \in P} \rho(x).$$

We are now in a position to define the generating function for μ . Let P be a poset, the *characteristic polynomial with respect to ρ and m* is defined by

$$\chi(P, t) = \sum_{x \in P} \mu(x) t^{m - \rho(x)} \quad (1)$$

where m is some integer with $m \geq \rho(P)$.

Although most of the results concerning the characteristic polynomial we present in this paper will be true regardless of whether the poset is ranked or not, we may from time to time need to assume that the poset is ranked. In the case when ρ is the normal rank function and $m = \rho(P)$ we will use the name *classic characteristic polynomial* to distinguish from the more general definition.

Let us do an example and calculate the characteristic polynomial of an unranked poset. We will consider the *Tamari lattices* [5, 9], which will be denoted by T_n . One way to define T_n is as the set of parenthesizations of the word $x_1 x_2 \cdots x_{n+1}$ with ordering given by saying π is covered by σ if there exists subwords A, B , and C such that

$$\pi = \dots ((AB)C) \dots \quad \text{and} \quad \sigma = \dots (A(BC)) \dots$$

Figure 1 displays the Hasse diagrams for T_3 .

As one can see from the Hasse diagram, T_3 is not ranked. In order to calculate the characteristic polynomial for T_3 we need a function, ρ . We will use generalized rank which was introduced in [2]. To define generalized rank, let us set up some notation. The atom set of P will be denoted by $A(P)$. Additionally, given an $x \in P$ we will use A_x to denote the set of atoms below x in P . If (A_1, A_2, \dots, A_n) is an ordered partition of the atoms of P , the *generalized rank* of an element x is given by

$$\rho(x) = |\{i : A_i \cap A_x \neq \emptyset\}|. \quad (2)$$

In other words, $\rho(x)$ counts the number of blocks in the partition that x is above.

Returning to the T_3 example, let us partition the atoms into

$$A_1 = \{((x_1 x_2)(x_3 x_4))\}, A_2 = \{((x_1(x_2 x_3))(x_4))\}.$$

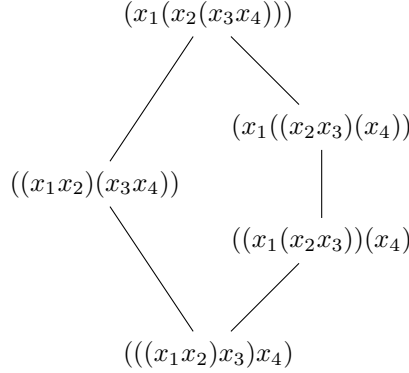


Figure 1: The Tamari Lattice T_3

Given this partition, we see that the generalized rank of the bottom element is 0, the three middle elements all have generalized rank 1 and the top element has generalized rank 2. We take $m = 3$ which is the length of the longest chain in T_3 . Using the definition of the characteristic polynomial (equation (1)), we get

$$\chi(T_3, t) = t^3 - 2t^2 + t = t(t-1)^2.$$

We see that $\chi(T_3, t)$ factors with roots 0 and 1. Recalling the well known fact that if P, Q are ranked then $\chi(P \times Q, t) = \chi(P, t)\chi(Q, t)$ one might ask if we can decompose T_3 into the product of two smaller posets. Using this reasoning, we might guess that T_3 is the product of two chains since chains have characteristic polynomials with roots 0 and 1. Of course, this cannot be the case since chains are ranked and so their products are too, but T_3 is not ranked. However, it is possible to take the product of the chains, collapse elements in the Hasse diagram without changing the characteristic polynomial and also get a poset isomorphic to T_3 .

For some posets the characteristic polynomial factors with nonnegative integer roots. In [8] a class of ranked lattices with this factorization was considered. We wish to give a generalization of these results to arbitrary finite posets with a minimum element. Many of the theorems from [8] still hold true at this level of generality, but we will need to develop some more concepts in order to show this.

In the next section, we review the idea of homogeneous quotient posets and how they apply to the Möbius function. Section 3 contains material about transversal functions and presents the first factorization theorem. We consider a specific type of transversal function in Section 4. This new type of function allows us to show another factorization theorem. We apply our factorization results to show a new family of lattices have characteristic polynomials which factor in section 5. We also prove Blass and Sagan's [2] result about LL lattices in this section. Section 6 is concerned with using quotient posets to derive some

classic results about the Möbius function. We finish with a section on future work.

2 Quotient Posets

We wish to order the classes of an equivalence relation on a poset. We recall some definitions from [8].

Definition 2.1. Let P be a poset and let \sim be an equivalence relation on P . We define the *quotient* P/\sim to be the set of equivalence classes with the binary relation \leq defined by $X \leq Y$ in P/\sim if and only if $x \leq y$ in P for some $x \in X$ and some $y \in Y$.

Quotients of posets are not necessarily posets. For example, take a 3-element chain and identify the bottom and top elements. The relation you obtain is reflexive and transitive, but not antisymmetric. In order to guarantee we get a poset when we take a quotient, we require two more properties.

Definition 2.2. Let P be a poset and let \sim be an equivalence relation on P . Order the equivalence classes as in the previous definition. We say the poset P/\sim is a *homogeneous quotient* if

- (1) $\hat{0}$ is in an equivalence class by itself, and
- (2) if $X \leq Y$ in P/\sim , then for all $x \in X$ there is a $y \in Y$ such that $x \leq y$.

It was shown in [8] that homogeneous quotients of finite posets are posets. Moreover, it was also shown how the Möbius function behaved when taking quotients. We describe this next.

We say that a homogeneous quotient P/\sim satisfies the *summation condition* if for all nonzero $X \in P/\sim$,

$$\sum_{y \in L(X)} \mu(y) = 0 \tag{3}$$

where $L(X)$ is the lower order ideal generated by X in P . This definition leads us to our first lemma.

Lemma 2.3 ([8]). *Let P/\sim be a homogeneous quotient poset which satisfies the summation condition. Then, for all equivalence classes X*

$$\mu(X) = \sum_{x \in X} \mu(x).$$

Not only will this lemma allow us to prove results about the factorization of the characteristic polynomial, we will also be able to use it to prove some classic results about the Möbius function. We will first consider the factorization theorems.

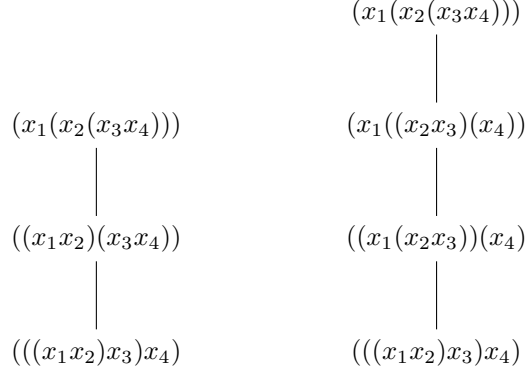


Figure 2: The Rooted Trees RT_{S_1} and RT_{S_2}

3 Transversal Functions

We begin this section by reviewing the notion of a rooted tree which was used in [8].

Definition 3.1. Let P be a poset and S be a subset of P which contains $\hat{0}$. Let \mathcal{C} be the collection of saturated chains of P which start at $\hat{0}$ and use only elements of S . The *rooted tree with respect to S* is the poset obtained by ordering \mathcal{C} by containment and will be denoted by RT_S .

First, let us note that a rooted tree contains a minimum element corresponding to the $\hat{0}$ - $\hat{0}$ chain. Additionally, there are no cycles in the Hasse diagram of a rooted tree. These two properties motivate the name for the poset. It also implies that only the minimum element and the atoms of the rooted tree have nonzero Möbius values.

By definition, if $x \in RT_S$, then x is a chain of the original poset. However, it will be useful to think of x as just the top element of the chain. That is, we think of x as just an element of the original poset.

Let us do an example of constructing rooted trees. Consider the Tamari lattice, T_3 , that was shown previously in Figure 1. We will take S_1 to be the upper order ideal generated by $((x_1x_2)(x_3x_4))$ together with $\hat{0}$ and S_2 to be the upper order ideal generated by $((x_1(x_2x_3))(x_4))$ together with $\hat{0}$. This gives RT_{S_1} and RT_{S_2} as shown in Figure 2.

To explain factorization of the characteristic polynomial of a poset, we will take products of rooted trees and then take a quotient of this product. We will denote elements of the product using boldface. So if S_1, S_2, \dots, S_n is a collection of subsets which contain $\hat{0}$, then a typical element of $\prod_{i=1}^n RT_{S_i}$ will be denoted $\mathbf{t} = (t_1, t_2, \dots, t_n)$.

When the poset is a lattice there is a canonical choice for the equivalence relation called the *standard equivalence relation* which was introduced in [8].

Since we are interested in posets which are not necessarily lattices we need to generalize this idea. To do this, we quotient out by the kernel of a special type of map from the product of rooted trees to the poset.

Definition 3.2. Let P be a poset and let (S_1, S_2, \dots, S_n) be an ordered collection of subsets of P each containing $\hat{0}$. We say $f : \prod_{i=1}^n RT_{S_i} \rightarrow P$ is a *transversal function* if it has the following properties:

1. The function f is order preserving.
2. The function f is surjective.
3. If $f(\mathbf{t}) = \hat{0}$, then $t_i = \hat{0}$ for all i .

If f is a transversal function, the *kernel of f* , denoted $\ker f$, is the equivalence relation \sim given by $\mathbf{s} \sim \mathbf{t}$ if and only if $f(\mathbf{s}) = f(\mathbf{t})$. Since we will often be referring to equivalence classes and the elements of these classes we need names for these objects.

Definition 3.3. Let P be a poset and let (S_1, S_2, \dots, S_n) be an ordered collection of subsets of P . Let $f : \prod_{i=1}^n RT_{S_i} \rightarrow P$ be a transversal function. If $\mathbf{t} \in \prod_{i=1}^n RT_{S_i}$ then we say \mathbf{t} is a *transversal* for x if $f(\mathbf{t}) = x$. We say \mathbf{t} is *atomic* or an *atomic transversal* if all the elements of \mathbf{t} are atoms of RT_{S_i} or $\hat{0}$. The set of all transversals for x will be denoted by \mathcal{T}_x and the set of all atomic transversals will be denoted by \mathcal{T}_x^a . We also define the *support* of a transversal, \mathbf{t} , as

$$\text{supp } \mathbf{t} = \{i : t_i \neq \hat{0}\}.$$

From the definitions it is evident that the set of equivalence classes of $\prod_{i=1}^n RT_{S_i} / \ker f$ is $\{\mathcal{T}_x : x \in P\}$. Moreover, it is clear that the size of the support of an atomic transversal for x is also its rank in the product of the rooted trees.

We are now in a position to give our first factorization theorem. The other factorization result we provide later will be a special case of this one.

Theorem 3.4. Let P be a poset with $\rho : P \rightarrow \mathbb{N}$ and let $m \in \mathbb{N}$ such that $\rho(P) \leq m$. Moreover, let (S_1, S_2, \dots, S_n) be an ordered collection of subsets of P which contain $\hat{0}$ and let f be a transversal function. Suppose the following hold.

- (1) If $x \leq y$ and $\mathbf{s} \in \mathcal{T}_x$, there exists $\mathbf{t} \in \mathcal{T}_y$ with $\mathbf{s} \leq \mathbf{t}$.
- (2) If $\mathbf{t} \in \mathcal{T}_x^a$, then $|\text{supp } \mathbf{t}| = \rho(x)$.
- (3) The summation condition (3) holds for all \mathcal{T}_x .

We can conclude the following.

- (a) We have an isomorphism

$$P \cong \left(\prod_{i=1}^n RT_{S_i} \right) / \ker f.$$

(b) For each $x \in P$,

$$\mu(x) = (-1)^{\rho(x)} |\mathcal{T}_x^a|.$$

(c) The characteristic polynomial of P with respect to ρ and m (equation (1)) is given by

$$\chi(P, t) = t^{m-n} \prod_{i=1}^n (t - |A(RT_{S_i})|).$$

Proof. First, we need to show the quotient is a homogeneous quotient. Conditions (2) and (3) in the definition of a transversal function (Definition 3.2) imply condition (1) of a homogeneous quotient (Definition 2.2). To show condition (2) holds, suppose that $\mathcal{T}_x \leq \mathcal{T}_y$. Then there is a $\mathbf{q} \in \mathcal{T}_x$ and a $\mathbf{r} \in \mathcal{T}_y$ with $\mathbf{q} \leq \mathbf{r}$. Since f is order preserving, $x = f(\mathbf{q}) \leq f(\mathbf{r}) = y$. By assumption (1) of the theorem, given a $\mathbf{s} \in \mathcal{T}_x$ there is a $\mathbf{t} \in \mathcal{T}_y$ with $\mathbf{s} \leq \mathbf{t}$ and so condition (2) of a homogeneous quotient is satisfied.

Now we show (a). Let $\bar{f} : (\prod_{i=1}^n RT_{S_i}) / \ker f \rightarrow P$ be the induced quotient map sending \mathcal{T}_x to x . Since f is surjective, it follows easily that \bar{f} is a bijection and so has an inverse say g .

Next we show \bar{f} is order preserving. Recall that the elements of the quotient, $(\prod_{i=1}^n RT_{S_i}) / \ker f$, are of the form \mathcal{T}_x for some $x \in P$. Suppose that $\mathcal{T}_x \leq \mathcal{T}_y$. Then again, since f is order preserving, $x \leq y$ and so \bar{f} is order preserving.

To finish the proof of (a), we show g is order preserving. Suppose that $x \leq y$. Since f is surjective, $\mathcal{T}_x \neq \emptyset$. Therefore, by assumption (1), there are $\mathbf{s} \in \mathcal{T}_x$ and $\mathbf{t} \in \mathcal{T}_y$ with $\mathbf{s} \leq \mathbf{t}$. Using the definition of a quotient poset, we get that that $\mathcal{T}_x \leq \mathcal{T}_y$ and so $g(x) \leq g(y)$.

Now we verify (b). By Lemma 2.3, assumption (3), and the fact that isomorphisms preserve Möbius values, we have that

$$\mu(x) = \sum_{\mathbf{t} \in \mathcal{T}_x} \mu(\mathbf{t}).$$

Since only atomic transversals have nonzero Möbius value we have

$$\mu(x) = \sum_{\mathbf{t} \in \mathcal{T}_x^a} \mu(\mathbf{t}).$$

By assumption (2), all the atomic transversals have the same support size which is the rank of x . It follows that each atomic transversal for x has Möbius value $(-1)^{\rho(x)}$. Therefore we have that

$$\mu(x) = (-1)^{\rho(x)} |\mathcal{T}_x^a|.$$

Finally we show (c). By definition,

$$\chi(P, t) = \sum_{x \in P} \mu(x) t^{m-\rho(x)}.$$

Using part (b), we get

$$\chi(P, t) = \sum_{x \in P} (-1)^{\rho(x)} |\mathcal{T}_x^a| t^{m-\rho(x)}.$$

We can break this sum into parts, depending on the rank of x . Note that by assumption (2) and part (b), every element with rank larger than n has Möbius value zero. Thus we have,

$$\chi(P, t) = \sum_{k=0}^n \left(\sum_{\rho(x)=k} (-1)^k |\mathcal{T}_x^a| t^{m-k} \right).$$

Neither $(-1)^k$ nor t^{m-k} depend on x so we can pull them out to get,

$$\chi(P, t) = \sum_{k=0}^n (-1)^k t^{m-k} \left(\sum_{\rho(x)=k} |\mathcal{T}_x^a| \right).$$

Using assumption (2) and denoting the k^{th} elementary symmetric function as e_k , we have the inner sum is exactly $e_k(|A(RT_{S_1})|, |A(RT_{S_2})|, \dots, |A(RT_{S_n})|)$. It follows that,

$$\chi(P, t) = \sum_{k=0}^n (-1)^k e_k(|A(RT_{S_1})|, |A(RT_{S_2})|, \dots, |A(RT_{S_n})|) t^{m-k}.$$

Pulling out a factor of t^{m-n} permits us to rewrite the sum as a product

$$\chi(P, t) = t^{m-n} \prod_{i=1}^n (t - |A(RT_{S_i})|)$$

completing the proof. □

4 Complete Transversal Functions

By definition, a transversal function must be surjective. However, if we impose more structure on the choice of subsets used to build the rooted trees, we can remove this assumption. In order to show this, we begin with a definition.

Definition 4.1. Let P be a poset and let A be a set of atoms. The *complete tree* (with respect to A) is the rooted tree $RT_{\hat{U}(A)}$ where $\hat{U}(A)$ is the upper order ideal generated by the set A together with $\hat{0}$.

Along with this new definition, we have a new type of function.

Definition 4.2. Let P be a poset and let (A_1, A_2, \dots, A_n) be an ordered partition of $A(P)$. We say $f : \prod_{i=1}^n RT_{\hat{U}(A_i)} \rightarrow P$ is a *complete transversal function* if it is order preserving and has the property that if in \mathbf{t} we have $t_i = \hat{0}$ or $t_i = x$ for all i , then $f(\mathbf{t}) = x$.

Note that it may appear that complete transversal functions are not transversal functions because we dropped the condition that they are surjective. However, we will see in the next lemma that, among other nice properties, the surjectivity of the function is a consequence of the definition. We also note that if we have a lattice, then $f(\mathbf{t}) = \vee \mathbf{t}$ is a complete transversal function where $\vee \mathbf{t} = t_1 \vee \dots \vee t_n$.

It will be useful to have notation for a new transversal obtained by inserting an element into a preexisting transversal. To do this, we will use $\mathbf{t}(e^i)$ to denote the transversal which is obtained by replacing the i^{th} coordinate of $\mathbf{t} = (t_1, t_2, \dots, t_n)$ with an element e . So we have,

$$\mathbf{t}(e^i) = (t_1, t_2, \dots, t_{i-1}, e, t_{i+1}, \dots, t_n).$$

Lemma 4.3. *Let P be a poset and let (A_1, A_2, \dots, A_n) be an ordered partition of $A(P)$. Let f be a complete transversal function. Then we can conclude the following.*

- (a) *The function f is surjective and f is a transversal function.*
- (b) *For all j , $t_j \leq f(t_1, t_2, \dots, t_n)$.*
- (c) *If $x \leq y$ and $\mathbf{s} \in \mathcal{T}_x$, there exists $\mathbf{t} \in \mathcal{T}_y$ with $\mathbf{s} \leq \mathbf{t}$.*
- (d) *For $x \in P$, let N_i be the number of atoms below x in A_i , then*

$$\sum_{\mathbf{s} \in L(\mathcal{T}_x)} \mu(\mathbf{s}) = \prod_{i=1}^n (1 - N_i). \quad (4)$$

- (e) *The summation condition (3) holds for all \mathcal{T}_x if and only if for all nonzero $x \in P$, there is an index i such that $|A_i \cap A_x| = 1$.*

Proof. First we show (a). Let $\hat{\mathbf{0}}$ be the transversal having all components equal to $\hat{0}$. Since we are using complete trees and a partition of the atom set, for every $x \in P$ there exists some i such that $\hat{\mathbf{0}}(x^i)$ is a transversal. It follows from the definition of a complete transversal function that $f(\hat{\mathbf{0}}(x^i)) = x$ and so f is surjective.

To show that the third condition for a transversal function holds, suppose that $f(\mathbf{t}) = \hat{0}$. By definition of a complete transversal function, $f(\hat{\mathbf{0}}(t_i^i)) = t_i$. Since f is order preserving and $\hat{\mathbf{0}}(t_i^i) \leq \mathbf{t}$ we get that $t_i = f(\hat{\mathbf{0}}(t_i^i)) \leq f(\mathbf{t}) = \hat{0}$. Therefore, if $f(\mathbf{t}) = \hat{0}$, then $\mathbf{t} = \hat{\mathbf{0}}$. This completes the proof that f is a transversal function.

For (b), we noted in the previous paragraph that

$$f(\hat{\mathbf{0}}(t_j^j)) = t_j.$$

Using the fact that f is order preserving, we get that

$$t_j = f(\hat{\mathbf{0}}(t_j^j)) \leq f(t_1, t_2, \dots, t_n).$$

Next we prove (c). This is trivial if $x = \hat{0}$ so assume x is nonzero. Let $\mathbf{s} \in \mathcal{T}_x$. Then by part (b), $s_i \leq x$ for all i . Let \mathbf{t} be given by $t_i = y$ for all i with $i \in \text{supp } \mathbf{s}$ and $t_i = \hat{0}$ for all other i . Such a \mathbf{t} is a valid transversal since $s_i \leq x \leq y$ and we are using complete trees. Note also that since $x \neq \hat{0}$ it must be that \mathbf{t} has at least one nonzero coordinate. It follows that $\mathbf{t} \in \mathcal{T}_y$ and $\mathbf{s} \leq \mathbf{t}$.

Next, let us show (d). We start by showing that

$$L(\mathcal{T}_x) = \{\mathbf{t} \text{ a transversal} : t_i \leq x \text{ for all } i\}. \quad (5)$$

To see that $L(\mathcal{T}_x)$ is contained in the other set, let $\mathbf{t} \in L(\mathcal{T}_x)$. Then for each i we have $\hat{\mathbf{0}}(t_i^i) \in L(\mathcal{T}_x)$. By definition of a complete transversal function, $f(\hat{\mathbf{0}}(t_i^i)) = t_i$. Since f is order preserving, $t_i = f(\hat{\mathbf{0}}(t_i^i)) \leq f(\mathbf{t}) = x$.

For the reverse inclusion, suppose that \mathbf{t} is a transversal with $t_i \leq x$ for all i . Let \mathbf{s} be the transversal obtained from \mathbf{t} by replacing all the nonzero t_i with x . We know that \mathbf{s} is a valid transversal because we are using complete trees. Since f is a complete transversal function, $f(\mathbf{s}) = x$ and so $\mathbf{s} \in L(\mathcal{T}_x)$. By construction, $\mathbf{t} \leq \mathbf{s}$ and therefore $\mathbf{t} \in L(\mathcal{T}_x)$.

Let I be the set of indices, i , such that there is an atom below x in A_i . By relabeling, if necessary, we may assume that $I = \{1, 2, \dots, j\}$. Since $N_i = 0$ implies that $1 - N_i = 1$,

$$\prod_{i=1}^j (1 - N_i) = \prod_{i=1}^n (1 - N_i).$$

From equation (5) we can conclude that the number of atomic transversals in $L(\mathcal{T}_x)$ with support size i is $e_i(N_1, N_2, \dots, N_j)$ where e_i is the i^{th} elementary symmetric function. Now for each atomic transversal $\mathbf{s} \in L(\mathcal{T}_x)$ we have that $\mu(\mathbf{s}) = (-1)^{|\text{supp}(\mathbf{s})|}$ and all other transversals have Möbius value zero. Therefore,

$$\sum_{\mathbf{s} \in L(\mathcal{T}_x)} \mu(\mathbf{s}) = \sum_{i=0}^j (-1)^i e_i(N_1, N_2, \dots, N_j) = \prod_{i=1}^j (1 - N_i)$$

which completes part (d).

Finally, (e) follows immediately from (d) and the definition of the summation condition. \square

Given this lemma, we can use Theorem 3.4 to immediately obtain the following.

Theorem 4.4. *Let P be a poset with $\rho : P \rightarrow \mathbb{N}$ and let $m \in \mathbb{N}$ such that $\rho(P) \leq m$. Moreover, let (A_1, A_2, \dots, A_n) be an ordered partition of $A(P)$ and let f be a complete transversal function. Suppose the following hold.*

- (1) *If $\mathbf{t} \in \mathcal{T}_x^a$, then $|\text{supp } \mathbf{t}| = \rho(x)$.*
- (2) *For all $x \in P$, there is an index i such that $|A_i \cap A_x| = 1$.*

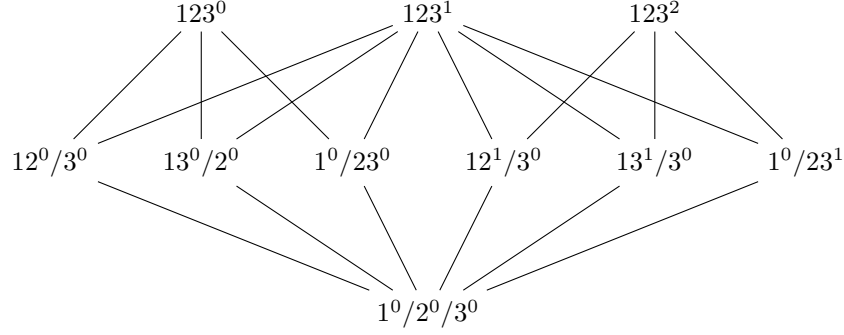


Figure 3: The Weighted Partition Poset Π_3^w

We can conclude the following.

(a) We have an isomorphism

$$P \cong \left(\prod_{i=1}^n RT_{\hat{U}(A_i)} \right) / \ker f.$$

(b) For each $x \in P$,

$$\mu(x) = (-1)^{\rho(x)} |\mathcal{T}_x^a|.$$

(c) The characteristic polynomial of P with respect to ρ and m is given by

$$\chi(P, t) = t^{m-n} \prod_{i=1}^n (t - |A_i|).$$

The reader may be wondering why we did not just assume from the start that we were using complete transversal functions. By doing so, we reduce the number of things we need to check and we still get the same conclusions as in Theorem 3.4. However, there are situations where the first theorem applies but the second does not.

Let us give an example where the summation condition (3) for \mathcal{T}_x needed in Theorem 3.4 holds, but the second condition of Theorem 4.4 does not. We will consider the weighted partition poset, Π_n^w introduced in [4]. The elements of Π_n^w are set partitions of $\{1, 2, \dots, n\}$ where each block B_i has one of the following weights $\{0, 1, \dots, |B_i| - 1\}$. The weighted partitions will be denoted by $B_1^{w_1}/B_2^{w_2}/\dots/B_n^{w_n}$ where w_i is the weight of block B_i . The ordering is given by

$$A_1^{v_1}/A_2^{v_2}/\dots/A_k^{v_k} \leq B_1^{w_1}/B_2^{w_2}/\dots/B_n^{w_n}$$

if and only if

1. We have

$$A_1/A_2/\dots/A_k \leq B_1/B_2/\dots/B_n$$

in the (unweighted) partition lattice Π_n .

2. If $B_l = A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_m}$, then

$$v_l - (w_{i_1} + w_{i_2} + \dots + w_{i_m}) \in \{1, 2, \dots, m-1\}.$$

The weighted partition poset Π_3^w is shown in Figure 3. It is easy to check that the classic characteristic polynomial of this poset factors as

$$\chi(\Pi_3^w, t) = (t-3)^2.$$

Consider the sets

$$A_1 = \{12^0/3^0, 13^0/2^0, 12^1/3^0, \hat{0}\}$$

and

$$A_2 = \{1^0/23^0, 13^1/3^0, 1^0/23^1, \hat{0}\}.$$

Additionally, consider the transversal function $f : \prod_{i=1}^2 RT_{A_i} \rightarrow \Pi_3^w$ which sends any pair which contains $\hat{0}$ to the other element in the pair and sends any pair with two non-zero elements to 123^i where i is the sum of their exponents. It is easy to check that f is a transversal function and that the summation condition (3) is satisfied. However, the element 123^1 is above every atom so it is impossible that it is above only one atom of either A_1 or A_2 . One can also check that all the conditions of Theorem 3.4 are satisfied and so we have verified that the classic characteristic polynomial does factor using our method.

We should also point out that, as was shown in [6], the classic characteristic polynomial of the weighted partition poset Π_n^w factors as $\chi(\Pi_n^w, t) = (t-n)^{n-1}$. This was shown using different methods than presented here. As of now, we do not have a transversal function which gives us the factorization.

5 Tamari Lattices and LL Lattices

Despite the reason explained earlier, Theorem 4.4 can be quite useful. First, we will show how to use it to explain the factorization of any interval of the Tamari lattice which implies a factorization result for both m -Tamari lattices, originally defined in [1], and the standard Tamari lattice. We note that these results are new. We will also use the theorem to give a nice formula for the Möbius function of intervals in the Tamari lattice. Finally, we will show that the theorem implies a result of Blass and Sagan [2] concerning LL lattices.

To show that the characteristic polynomial of the intervals of the Tamari lattices factor, we introduce a different way to denote the elements. For each element of $\sigma \in T_n$ we will give a corresponding *left-bracket vector* (v_1, v_2, \dots, v_n) . The value of v_i is obtained by locating x_i in σ then, moving left, counting the number of x 's (including x_i) and left parentheses that you pass until the two

numbers are the same. At this point, stop and set $v_i = j$ where x_j is the last x that was passed before the two numbers became equal.

Let us do an example of calculating a left-bracket vector. Suppose that we let $\sigma = ((x_1x_2)(x_3x_4))$, then to find v_1 look for x_1 and move left. Immediately to the left of x_1 we find a left parentheses and so $v_1 = 1$. Next, we have that $v_2 = 1$ since x_1 and x_2 are adjacent and are preceded by two left parentheses. Finally, for v_3 notice that just preceding x_3 we have a left parentheses and so $v_3 = 3$. Therefore, the left-bracket vector associated to $((x_1x_2)(x_3x_4))$ is $(1, 1, 3)$.

If we use left-bracket vectors, the partial order for the Tamari lattice is defined by $(v_1, v_2, \dots, v_n) \leq (w_1, w_2, \dots, w_n)$ provided $v_i \leq w_i$ for all $1 \leq i \leq n$. In addition to the simple way the partial order is defined using left-bracket vectors, the join operation also has a nice description.

Proposition 5.1 ([9]). *Using left-bracket vector notation, the join operation in the Tamari lattice is as follows,*

$$(v_1, v_2, \dots, v_n) \vee (w_1, w_2, \dots, w_n) = (\max(v_1, w_1), \max(v_2, w_2), \dots, \max(v_n, w_n)).$$

In the next proof and the sequel we will use the notation $x \leq y$ to indicate that y covers x .

Proposition 5.2. *Let T_n be the Tamari lattice and let I be any interval in T_n . Let ρ be generalized rank as defined by equation (2) and let m be the length of the longest chain in I . If there are k atoms in the interval I and $\chi(I, t)$ is the characteristic polynomial with respect to ρ and m , then*

$$\chi(I, t) = t^{m-k}(t-1)^k.$$

Proof. Partition the atoms of I as (A_1, A_2, \dots, A_k) where each A_k has exactly one atom and use the complete transversal function $f(\mathbf{t}) = \vee \mathbf{t}$. With this partition we trivially get condition (2) of Theorem 4.4. Since we are using generalized rank, we must show that the join of any j atoms is above exactly j atoms in order to show condition (1). We will use the left-bracket vector representation of the elements of the Tamari lattice to verify this.

If v and w are left-bracket vectors and $v \leq w$, then it is easy to see that v and w agree in all but one position. Additionally, if we take j atoms of the interval they all cover the same element in the Tamari lattice. It follows that each of the atoms differs from the $\hat{0}$ of the interval in one of j distinct positions. Using Proposition 5.1, we can see that the join of j atoms of the interval disagrees with the bottom element of the interval in exactly j places. Let $x = a_1 \vee a_2 \vee \dots \vee a_j$ where the $a_i \in A(I)$. Suppose that $b \in A(I)$ with $b \leq x$. Then b differs from the $\hat{0}$ of I in exactly one place. Moreover, since $b \leq x$, it must be one of the j positions where x disagrees with the $\hat{0}$ of I . This implies that $b = a_i$ for some i . Therefore, the join of j atoms is above exactly those j atoms.

Finally, we must show that $m \geq \rho(I)$ where m is the length of the longest chain in I since this was required in the definition of the characteristic polyno-

mial. Let x_0 be the $\hat{0}$ element of I and for each $1 \leq i \leq k$ define

$$x_i = \bigvee_{l=1}^i a_l$$

where a_i is the unique element of A_i . Since the join of j atoms is above exactly those j atoms, we know that all the x_i 's are distinct. It follows that I contains a chain of length k , namely the chain $x_0 < x_1 < \dots < x_k$. Since ρ is generalized rank and since we partitioned the atom set into k blocks, if m is the length of the largest chain in I then $\rho(I) = k \leq m$. Applying Theorem 4.4 now yields the result. \square

Let us discuss some consequences of this proposition. First, since the length of the longest chain in T_n is $\binom{n}{2}$ and this poset has $n - 1$ atoms, we get that

$$\chi(T_n, t) = t^{\binom{n-1}{2}}(t - 1)^{n-1}$$

which was originally shown in [2].

The other consequence concerns the factorization of the m -Tamari lattice. Fix an m and n , noting that m here is not being used as it was earlier in the paper. Following the definitions given in [3] an m -ballot path of size n is a path in the first quadrant of \mathbb{R}^2 from $(0, 0)$ to (mn, n) using unit steps north and east which never goes below the line $x = my$. Suppose that P is an m -ballot path with an east step E immediately followed by a north step N . Another path Q covers P if Q is obtained from P by switching E and S where S is the shortest factor of P which starts at N and is an m -ballot path. The set of m -ballot paths with this covering relation defines the m -Tamari lattice.

In [3, Proposition 4], it was shown that the m -Tamari lattices are isomorphic to intervals in the Tamari lattice. Therefore, we see that the characteristic polynomials of the m -Tamari lattices also have a nice factorization.

Since we verified the assumptions of Theorem 4.4 in the proof of Proposition 5.2, we can also give a characterization of the Möbius function of the intervals of the Tamari lattice. We explain this characterization for the full Tamari lattice, but there is a similar formula for the intervals. Write a left-bracket vector $v = (v_1, v_2, \dots, v_n)$ in multiplicity notation $1^{k_1} 2^{k_2} \dots n^{k_n}$ where k_i is the number of times that i appears as an entry in v .

In the proof of the following proposition we will make use of an equivalent definition of left-bracket vectors. As explained in [2] a vector, $v = (v_1, v_2, \dots, v_n)$, consisting of positive integers is a left-bracket vector if and only if the following hold.

1. For all i , $1 \leq v_i \leq i$.
2. Letting $S_i = \{v_i, v_i + 1, \dots, i\}$, for any S_i and S_j either $S_i \cap S_j = \emptyset$ or one set is contained in the other.

With this equivalent definition, we can now state and prove a result about the Möbius function of the Tamari lattice. Note the similarity of the Möbius

function of the Tamari lattice and the Möbius function of the divisor lattice in the following proposition.

Proposition 5.3. *Let T_n be the Tamari lattice. If $v = 1^{k_1} 2^{k_2} \dots n^{k_n}$ is written in multiplicity notation, then*

$$\mu(1^{k_1} 2^{k_2} 3^{k_3} \dots n^{k_n}) = \begin{cases} (-1)^{k_2+k_3+\dots+k_n} & \text{if } 2^{k_2} 3^{k_3} \dots n^{k_n} \text{ is square free,} \\ 0 & \text{otherwise,} \end{cases}$$

where square free means that $k_2, k_3, \dots, k_n \leq 1$.

Proof. Let $v = (v_1, v_2, \dots, v_n)$ be a left-bracket vector such that written in multiplicity notation $2^{k_2} 3^{k_3} \dots n^{k_n}$ is square free. We claim that in this case $v_j = 1$ or $v_j = j$ for all j . Suppose that this was not the case and let v_j be such that $v_j \neq 1, j$. This implies that $1 < v_j < j$. Additionally, $1 \leq v_{v_j} < v_j$ where the last inequality is strict since v is square free. So $S_{v_j} = \{v_{v_j}, v_{v_j} + 1, \dots, v_j\}$ and $S_j = \{v_j, v_j + 1, \dots, j\}$ with $|S_{v_j}|, |S_j| \geq 2$. Thus $S_{v_j} \cap S_j = \{v_j\} \neq \emptyset$ but neither set contains the other, which gives the desired contradiction.

In the proof of Proposition 5.2, we showed that the conditions of Theorem 4.4 hold if we use the partition of the atoms of T_n into singleton blocks. Written as left-bracket vectors, the atoms of T_n are of the form $(1, 1, \dots, i, \dots, 1)$ where i is in the i^{th} position. From conclusion (b) of Theorem 4.4, we know that the Möbius value of an element $v \in T_n$ is $(-1)^{\rho(v)} |\mathcal{T}_v^a|$. We showed that the join of j atoms is above exactly those j atoms and so each element has at most one atomic transversal. Using Proposition 5.1 and the previous paragraph we see that if the numbers $2, 3, \dots, n$ appear at most once in the left-bracket vector of v , then v has an atomic transversal. And, using the same proposition, if v has an atomic transversal containing j atoms then it is square free. So in this case $\mu(v) = (-1)^j$. The rest of the elements of T_n have no atomic transversal and so $\mu(v) = 0$ completing the proof of the proposition. \square

Having shown what Theorem 4.4 can say about the Tamari lattices, we now turn our attention to seeing how it implies a theorem of [2]. To explain both the theorem as well as how to prove it, we begin by defining the notion of a partition of an atom set being induced by a multichain.

Let P be a poset with $\hat{1}$ and let $C : \hat{0} = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = \hat{1}$ be a $\hat{0}$ - $\hat{1}$ multichain of P . We say (A_1, A_2, \dots, A_n) is *induced* by C if for all i ,

$$A_i = \{a \in A(P) \mid a \leq x_i \text{ and } a \not\leq x_{i-1}\}.$$

Although a partition induced by a multichain exists for any poset with a maximum element, for the next definition we will need to have a lattice.

Definition 5.4 ([8]). Let L be a lattice and let $C : \hat{0} = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = \hat{1}$ be a $\hat{0}$ - $\hat{1}$ multichain. For atomic $x \in L$, x neither $\hat{0}$ nor an atom, let i be the index such that $x \leq x_i$ but $x \not\leq x_{i-1}$. We say that C satisfies the *meet condition* if, for each such x , we have $x \wedge x_{i-1} \neq \hat{0}$.

It was shown in [8] that if a partition is induced by a multichain, then assumption (2) of Theorem 4.4 holds if and only if the multichain satisfies the meet condition. We will call a multichain, $C : \hat{0} = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = \hat{1}$, *saturated* if for every inequality $x_{i-1} \leq x_i$ either $x_{i-1} = x_i$ or $x_{i-1} \lhd x_i$.

Recall that an element x in a lattice L is called *left-modular* if for all $y, z \in L$ with $y \leq z$ we have the following equality

$$y \vee (x \wedge z) = (y \vee x) \wedge z.$$

We call a multichain *left-modular* if every element of the multichain is left-modular.

In [8] it was shown that saturated $\hat{0}$ – $\hat{1}$ left-modular chains satisfy the meet condition. If C is a chain which satisfies the meet condition and M is a multichain formed by using all the elements of C at least once, then it is not hard to see that M must also satisfy the meet condition. It follows that $\hat{0}$ – $\hat{1}$ saturated left-modular multichains satisfy the meet condition.

The authors in [8] used the fact that partitions induced by saturated $\hat{0}$ – $\hat{1}$ left-modular chains imply assumption (2) of Theorem 4.4 to prove Stanley's Supersolvability Theorem [11]. We will use this fact to prove Blass and Sagan's result about LL lattices [2] which is a generalization of the supersolvability result.

In order to explain this result, we need to define the level condition. Let (A_1, A_2, \dots, A_n) be induced by $C : \hat{0} = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = \hat{1}$. This multichain also induces a partial ordering on the atoms denoted by \triangleleft . It is defined by saying $a \triangleleft b$ if $a \in A_i$ and $b \in A_j$ with $i < j$. We say that a lattice L with chain C satisfies the *level condition* if

$$a \triangleleft b_1 \triangleleft b_2 \triangleleft \cdots \triangleleft b_k$$

implies that

$$a \not\leq \bigvee_{i=1}^k b_i.$$

The lattice L is called an *LL lattice* if it contains a left-modular multichain C and L together with C satisfy the level condition. We are now in a position to state Blass and Sagan's result.

Theorem 5.5 ([2]). *Let L be a lattice and let (A_1, A_2, \dots, A_n) be induced by a left-modular saturated multichain such that L is an LL lattice. Let ρ be generalized rank and let m be the length of the longest $\hat{0}$ – $\hat{1}$ chain. Then*

$$\chi(L, t) = t^{m-n} \prod_{i=1}^n (t - |A_i|).$$

Before we prove the theorem, let us note that it is possible to have $n > m$ in which case the exponent on the outside of the factorization will be negative. This is possible since we are using multichains and so repeating elements in the

chain will give rise to as many empty blocks in the partition of the atom set as we wish. However, for each such block, we get a corresponding factor $(t - 0)$. Thus $\chi(L, t)$ is still a polynomial since the negative power of t on the outside of the product will be canceled by the positive powers of t on the inside of the product.

Proof. We wish to use Theorem 4.4. First, note that since we are using generalized rank we have that $\rho(P)$ is at most the number of nonempty blocks in the partition. Since our partition is induced by a multichain and since m is the length of the largest chain in the lattice, we have that $\rho(P) \leq m$.

Define the complete transversal function to be $f(\mathbf{t}) = \vee \mathbf{t}$. Although it is not worded in the same way, the authors in [2, Theorem 6.3 and Lemma 6.4] proved assumption (1) of Theorem 4.4 holds. Finally, as noted before, it was shown in [8] that saturated left-modular multichains satisfy the meet condition and so satisfy assumption (2) of Theorem 4.4. \square

The theorems presented so far have provided conditions which imply factorization. We would like to finish this section with a theorem where we provide a condition which is equivalent to factorization.

Theorem 5.6. *Let P be a poset and let $\rho : P \rightarrow \mathbb{N}$ with $m \in \mathbb{N}$ such that $\rho(P) \leq m$. Let $\chi(P, t)$ be the characteristic polynomial with respect to ρ and m . Let (A_1, A_2, \dots, A_n) be an ordered partition of $A(P)$ and let $f : \prod_{i=1}^n RT_{\hat{U}(A_i)} \rightarrow P$ be a complete transversal function. Finally, define*

$$T = \{x \in P \setminus \hat{0} : |A_i \cap A_x| \neq 1 \text{ for all } i\}.$$

Suppose that the following hold.

1. *If $\mathbf{t} \in \mathcal{T}_x^a$ then $|\text{supp}(\mathbf{t})| = \rho(x)$.*
2. *If $x, y \in P$ and $x < y$, then $\rho(x) < \rho(y)$.*
3. *For all minimal elements $x, y \in T$, the cardinality of the sets*

$$\{i : |A_i \cap A_x| \neq 0\} \text{ and } \{i : |A_i \cap A_y| \neq 0\}$$

have the same parity.

Under these conditions,

$$\chi(P, t) = t^{m-n} \prod_{i=1}^n (t - |A_i|)$$

if and only if for every nonzero $x \in P$ there is an index i such that $|A_i \cap A_x| = 1$.

This theorem is a generalization of Theorem 17 shown in [8]. The two proofs are quite similar so we only provide a sketch below.

Sketch of proof. First, note that the backwards direction is Theorem 4.4. For the forward direction, we will prove the contrapositive. Note that the assumption in this direction implies that $T \neq \emptyset$. Let k be the smallest value of ρ applied to the elements of T . We show that the coefficient of t^{m-k} in $\chi(P, t)$ and in $t^{m-n} \prod_{i=1}^n (t - |A_i|)$ are different.

Define $R = \left(\prod_{i=1}^n RT_{\hat{U}(A_i)} \right) / \ker f$. We claim that R is a homogeneous quotient and that $P \cong R$. Since f is a complete transversal function, Lemma 4.3 part (c) implies that assumption (1) of Theorem 3.4 is satisfied. Note that the proof of part (a) of Theorem 3.4 only requires assumption (1). Therefore, R is homogeneous and $P \cong R$. Since $P \cong R$, it is enough to show that the coefficient of t^{m-k} in $\chi(R, t)$ and in $t^{m-n} \prod_{i=1}^n (t - |A_i|)$ are not the same.

Let x_1, x_2, \dots, x_l be the set of elements of T with $\rho(x_i) = k$ for all i and let $S = \{\mathcal{T}_{x_1}, \mathcal{T}_{x_2}, \dots, \mathcal{T}_{x_l}\}$ be the corresponding equivalence classes. Moreover, define Q to be the poset obtained from R by removing all elements of R with ρ value larger than k . Using assumption (2), we can see that the Möbius value of elements with ρ at most k in R and Q are the same. In Q all elements with ρ value k are maximal. By assumption (2) and the assumption on k any element of Q which is not maximal cannot be in the set T . Then Lemma 4.3 part (e) implies that every non-maximal element satisfies the summation condition (3). Thus, we can apply Lemma 14 in [8] to conclude that

$$\mu(\mathcal{T}_{x_i}) = \sum_{\mathbf{t} \in \mathcal{T}_{x_i}} \mu(\mathbf{t}) - \sum_{\mathbf{s} \in L(\mathcal{T}_{x_i})} \mu(\mathbf{s}).$$

Let

$$c_i = \sum_{\mathbf{s} \in L(\mathcal{T}_{x_i})} \mu(\mathbf{s}).$$

We claim that all the c_i 's are either 0 or have the same sign. By equation (4), if $c_i \neq 0$, then the sign of c_i is $(-1)^{k_i}$ where k_i is the number of blocks with atoms below x_i . By assumption (3), for the c_i 's which are not equal to 0, the corresponding k_i 's have the same parity. Therefore, the signs of the nonzero c_i 's are the same. Since there is at least one element of $T \neq \emptyset$ in Q , there is at least one $c_i \neq 0$.

Using the same argument as in the proof of Theorem 17 of [8], we get the coefficient of t^{m-k} in $\chi(R, t)$ is

$$\sum_{|\text{supp } \mathbf{t}|=k} \mu(\mathbf{t}) - \sum_{i=1}^l c_i$$

in which the first sum ranges over atomic transversals. Since there is at least one c_i which is nonzero and all the nonzero c_i 's have the same sign, we see that this coefficient is not the same as

$$\sum_{|\text{supp } \mathbf{t}|=k} \mu(\mathbf{t}).$$

However, the previous expression is the coefficient of t^{m-k} in $t^{m-n} \prod_{i=1}^n (t - |A_i|)$. It follows that

$$\chi(P, t) \neq t^{m-n} \prod_{i=1}^n (t - |A_i|)$$

which is what we wished to show. \square

6 Classic Results About the Möbius Function

In this section, we will give a new method to prove an array of classic results about the Möbius function. The idea of the method is to use induction on the size of the poset. In order to do this, we will collapse a coatom and the $\hat{1}$ of the poset.

We begin with a lemma that explains the simple nature of the values of μ for the original poset and the poset obtained by collapsing a coatom and $\hat{1}$. In the lemma and throughout the rest of the section, we will use $[x]$ to denote the equivalence class which contains x .

Lemma 6.1. *Let P be a poset with a $\hat{0}$ and $\hat{1}$ and at least 3 elements. Suppose c is a coatom and let \sim be the equivalence relation identifying c and $\hat{1}$. Then P/\sim is homogeneous and*

$$\mu([\hat{1}]) = \mu(c) + \mu(\hat{1}).$$

Moreover, if P is a lattice, then P/\sim is a lattice with $[x] \vee [y] = [x \vee y]$ for all $x, y \in P$ and $[x] \wedge [y] = [x \wedge y]$ provided $[x], [y] \neq [\hat{1}]$.

Proof. First, let us show that P/\sim is homogeneous. Since there are at least 3 elements and we are collapsing a coatom and $\hat{1}$, we have that $\hat{0}$ is in its own equivalence class. Now suppose that $[x] < [y]$. It follows that $[x] \neq \{c, \hat{1}\}$ since $[x] < [y]$ and $[c] = [\hat{1}]$ is the $\hat{1}$ of the quotient. Therefore, $[x] = \{x\}$ and so it is obvious that P/\sim is a homogeneous quotient.

To show that $\mu([\hat{1}]) = \mu(c) + \mu(\hat{1})$ note that since every element of P is below $\hat{1}$ and every other equivalence class has only one element, we get

$$\sum_{y \in L([x])} \mu(y) = \sum_{y \leq x} \mu(y) = 0$$

for all nonzero $x \neq c$. By Lemma 2.3 this implies that

$$\mu([\hat{1}]) = \mu(c) + \mu(\hat{1})$$

which is what we wished to prove.

Now suppose that P is a lattice. It is not hard to see that $(P/\sim) \cong (P \setminus \{c\})$. Therefore, if $x \vee y \neq c$, we immediately get that $[x] \vee [y]$ exists and $[x] \vee [y] = [x \vee y]$. If $x \vee y = c$, then $\hat{1}$ is the only element in $P \setminus \{c\}$ which is an upper bound for both x and y . It follows that $[x] \vee [y] = [\hat{1}] = [c] = [x \vee y]$. Since $P \setminus \{c\}$ clearly has a $\hat{0}$, we conclude P/\sim is a lattice. Finally, if $[x], [y] \neq [\hat{1}]$ then $[x] = \{x\}$ and $[y] = \{y\}$ and so $[x] \wedge [y] = [x \wedge y]$. \square

Let us now use Lemma 6.1 to prove some classic results.

Corollary 6.2 (Hall's Theorem [7]). *Let P be a finite poset, then*

$$\mu(x, y) = \sum_{i \geq 0} (-1)^i c_i$$

where c_i is the number of chains of length i which start at x and terminate at y .

Proof. Without loss of generality we may assume that $x = \hat{0}$ and $y = \hat{1}$ since all chains which start at x and terminate at y are in the interval $[x, y]$. We prove the theorem by inducting on $|P|$. If $|P| = 1$ or $|P| = 2$ then the result is obvious.

Now suppose that $|P| > 2$. Let P/\sim be obtained by identifying a coatom c and $\hat{1}$. Consider the sum

$$\sum_{i \geq 0} (-1)^i c_i$$

where c_i is the number of $\hat{0}$ – $\hat{1}$ chains of length i in P . Let a_i be the number chains of length i which do not contain c and let b_i be the number chains of length i containing c . Then

$$\sum_{i \geq 0} (-1)^i c_i = \sum_{i \geq 0} (-1)^i a_i + \sum_{i \geq 0} (-1)^i b_i.$$

There exists a bijection between $\hat{0}$ – $\hat{1}$ chains in P not containing c and $[\hat{0}]$ – $[\hat{1}]$ chains in P/\sim which preserves length. Moreover, there is a bijection between $\hat{0}$ – $\hat{1}$ chains in P containing c and $\hat{0}$ – c chains in $[0, c]$. Note that in this bijection, the chains decrease by one in length.

Since $|P/\sim| < |P|$, using induction we get that

$$\mu([\hat{1}]) = \sum_{i \geq 0} (-1)^i a_i.$$

Similarly since $|[0, c]| < |P|$ we get that

$$\mu(c) = - \sum_{i \geq 0} (-1)^i b_i$$

where we have multiplied the sum by -1 since the chains have decreased by one in length.

By Lemma 6.1, we have that

$$\mu([\hat{1}]) = \mu(c) + \mu(\hat{1})$$

or equivalently

$$\mu(\hat{1}) = \mu([\hat{1}]) - \mu(c).$$

Therefore,

$$\mu(\hat{1}) = \sum_{i \geq 0} (-1)^i a_i + \sum_{i \geq 0} (-1)^i b_i = \sum_{i \geq 0} (-1)^i c_i$$

which is what we wished to prove. \square

Next, we prove a theorem of Weisner.

Corollary 6.3 (Weisner's Theorem [12]). *Let L be a lattice and let $\hat{0} \neq a \in L$. If $|L| \geq 2$, then*

$$\mu(\hat{1}) = - \sum_{x \neq \hat{1}, x \vee a = \hat{1}} \mu(x).$$

Proof. Let us note that if $a = \hat{1}$, then the result is just restating the definition of μ , so we assume that $a \neq \hat{1}$ for the rest of the proof. We prove the result by induction. We have already covered the case $|L| = 2$, since then a must be $\hat{1}$.

Now suppose that $|L| > 2$. Let c be a coatom such that $a \leq c$. Consider, the lattice L/\sim obtained by identifying c and $\hat{1}$. Since $|L/\sim| < |L|$, we get that

$$\mu([\hat{1}]) = - \sum_{[x] \neq [\hat{1}], [x] \vee [a] = [\hat{1}]} \mu([x]).$$

Using the facts that $[\hat{1}] = \{c, 1\}$, $[x] \vee [a] = [x \vee a]$, and $\mu([x]) = \mu(x)$ for $[x] \neq [\hat{1}]$, we obtain,

$$\mu([\hat{1}]) = - \sum_{x \neq c, \hat{1}, x \vee a = c, \hat{1}} \mu(x).$$

Since joins are unique, we can break the sum into two parts as,

$$\mu([\hat{1}]) = - \sum_{x \neq c, \hat{1}, x \vee a = c} \mu(x) - \sum_{x \neq c, \hat{1}, x \vee a = \hat{1}} \mu(x).$$

If $x \vee a = c$, then it is clear that $x \in [0, c]$. Moreover, since $a \leq c$, it is clear that $c \vee a \neq \hat{1}$. Thus, we can remove the $x \neq \hat{1}$ condition in the first sum and remove the $x \neq c$ condition in the second. This gives,

$$\mu([\hat{1}]) = - \sum_{x \neq c, x \vee a = c} \mu(x) - \sum_{x \neq \hat{1}, x \vee a = \hat{1}} \mu(x).$$

Now the first sum is only over $[0, c]$ and $|[0, c]| < |L|$ so by induction,

$$\mu([\hat{1}]) = \mu(c) - \sum_{x \neq \hat{1}, x \vee a = \hat{1}} \mu(x).$$

Using the fact that $\mu([\hat{1}]) = \mu(\hat{1}) + \mu(c)$, we immediately obtain the result. \square

Our next corollary will make use of crosscuts. We remind the reader of the definition here.

Definition 6.4. Let L be a lattice. A *crosscut* of L is a set C with the following properties:

1. $\hat{0}, \hat{1} \notin C$.
2. C is an antichain.

3. Every maximal $\hat{0}$ - $\hat{1}$ chain intersects C .

Theorem 6.5 (Rota's Crosscut Theorem [10]). *Let L be a lattice and let C be a crosscut. Then*

$$\mu(\hat{1}) = \sum_{\vee B = \hat{1}, \wedge B = \hat{0}} (-1)^{|B|}$$

where the sum ranges over all $B \subseteq C$ such that $\vee B = \hat{1}$ and $\wedge B = \hat{0}$.

Proof. We first consider the special case when every coatom is also an atom. In this case, the crosscut must be the atom set. Moreover, a subset of the crosscut has meet $\hat{0}$ and join $\hat{1}$ if and only if it has at least two elements. Therefore, if L has n atoms we obtain the following

$$\sum_{\vee B = \hat{1}, \wedge B = \hat{0}} (-1)^{|B|} = \sum_{|B| \geq 2} (-1)^{|B|} = \sum_{k=2}^n (-1)^k \binom{n}{k} = n - 1.$$

This agrees with the value of $\mu(\hat{1})$ when L has n atoms and every coatom is an atom. Thus, the result holds in this special case.

Recall that if L^* is the dual lattice of L , then $\mu_L(\hat{1}) = \mu_{L^*}(\hat{1})$. Moreover, in L^* , joins and meets reverse roles. Therefore, if we have a crosscut consisting of only coatoms, then we can consider the dual lattice. As a result, we may now assume that there is always at least one coatom in the lattice which is not in the crosscut. With this in mind we proceed by induction on $|L|$.

If $|L| \leq 3$, then it must be that $|L| = 3$ since smaller lattices do not have crosscuts. We have already done the case when $|L| = 3$. Suppose that $|L| > 3$ and let c be a coatom that is not in the crosscut. Consider the lattice L/\sim where we collapse c and $\hat{1}$. Since c was not in the crosscut we still have the same crosscut. By induction, we know that

$$\mu([\hat{1}]) = \sum_{\vee B = [\hat{1}], \wedge B = [\hat{0}]} (-1)^{|B|}.$$

Lemma 6.1 implies that $\vee B = [\hat{1}]$ in L/\sim if and only if $\vee B = c$ or $\vee B = \hat{1}$ in L . Additionally, since C does not contain c nor $\hat{1}$, Lemma 6.1 also implies that $\wedge B = [\hat{0}]$ in L/\sim if and only if $\wedge B = \hat{0}$ in L . Therefore, we can break the previous sum as follows

$$\mu([\hat{1}]) = \sum_{\vee B = c, \wedge B = \hat{0}} (-1)^{|B|} + \sum_{\vee B = \hat{1}, \wedge B = \hat{0}} (-1)^{|B|}.$$

Note that if $\vee B = c$, then B must only have elements in $[\hat{0}, c]$. Thus the first sum in the previous equation is over B contained in $[\hat{0}, c] \cap C$ such that $\vee B = c$ and $\wedge B = \hat{0}$. Since $||[\hat{0}, c]| < |L|$, induction implies that

$$\mu([\hat{1}]) = \mu(c) + \sum_{\vee B = \hat{1}, \wedge B = \hat{0}} (-1)^{|B|}.$$

Subtracting $\mu(c)$ from both sides and applying Lemma 6.1 we see that

$$\mu(\hat{1}) = \sum_{\vee B = \hat{1}, \wedge B = \hat{0}} (-1)^{|B|}$$

which completes the proof. \square

7 Future Work

We saw in Proposition 5.3 that the Möbius function for the Tamari lattice has a description which is similar to that of the divisor lattice. In the future, we hope that we can show this fact by exhibiting a quotient of the divisor lattice which preserves the Möbius function. More generally, it would be nice to find other examples of unranked posets whose Möbius function behaves like that of a ranked poset.

It is easy to see that given a partition of the atom set of a poset and an element x , the elements of the set \mathcal{T}_x can be viewed as the facets of an abstract simplicial complex $\Delta(\mathcal{T}_x)$. This complex encodes some useful information. For example, if $\mathbf{t} \in \mathcal{T}_x^a$ then as an element of the product of rooted trees $\mu(\mathbf{t}) = (-1)^{|\text{supp } \mathbf{t}|}$. Therefore

$$\sum_{\mathbf{t} \in L(\mathcal{T}_x)} \mu(\mathbf{t}) = \sum_{\mathbf{t} \in L(\mathcal{T}_x^a)} (-1)^{|\text{supp } \mathbf{t}|}.$$

Since the dimension of \mathbf{t} in $\Delta(\mathcal{T}_x)$ is $|\text{supp } \mathbf{t}| - 1$ the previous equation can be rewritten as

$$\sum_{\mathbf{t} \in L(\mathcal{T}_x)} \mu(\mathbf{t}) = \sum_{\mathbf{t} \in L(\mathcal{T}_x^a)} (-1)^{\dim \mathbf{t} + 1}.$$

This is equivalent to

$$\sum_{\mathbf{t} \in L(\mathcal{T}_x)} \mu(\mathbf{t}) = - \sum_{\mathbf{t} \in L(\mathcal{T}_x^a)} (-1)^{\dim \mathbf{t}}.$$

One can see that the right-hand side of the previous equation is the negative of the reduced Euler characteristic of $\Delta(\mathcal{T}_x^a)$. It follows that the summation condition (3) is satisfied if and only if for each x the reduced Euler characteristic of $\Delta(\mathcal{T}_x^a)$ is $-\delta_{\hat{0}, x}$. We also note that if we restrict to the set of atomic transversals for x , $\Delta(\mathcal{T}_x^a)$ is pure of dimension $\rho(x) - 1$ if and only if condition (2) of Theorem 3.4 holds. We are interested in investigating what else this complex can tell us about the poset.

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